Math 206A Lecture 18 Notes

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1 Meditations on Cauchy's Theorem

1.1 Alexandor's theorem and Stoker's Conjecture

Theorem 1.1 (Alexandor, 1920s). *Let $P, P' \subseteq \mathbb{R}^3$ are convex polytopes with $\Phi : \alpha(P) \to \alpha(P')$ is such that for a;; $F \in \alpha(P)$ with $\dim(F) = 2$, $\{\angle \text{ in } F\} \simeq \{\angle \text{ in } \Phi(F)\}$. Then P and P' have equal corresponding dihedral angles.*

This is really a corollary of our proof of Cauchy's theorem. We basically proved this as a lemma to get Cauchy's theorem.

Here is a related conjecture.

Theorem 1.2 (Stoker's conjecture, 1960s). *If you know all face angles, you know all dihedral angles and vice versa.*

People believe this to be true, but the conjecture is still open.

1.2 Non-examples to Cauchy's theorem

Here are some non-examples of Cauchy's theorem.

Example 1.1. Take a triangular prism, and remove a triangular pyramid from one of the sides. This is not convex, so Cauchy's theorem doesn't apply, even though it has the same lattice as the triangular prism with with a triangular pyramid on top. But we can get from one to another by continuously deforming.

Corollary 1.1 (Cauchy). *Let $\{P_t : t \in [0, 1]\}$ be a continuous family of 3-dimensional convex polytopes such that $\alpha(P_t) \cong\simeq \alpha(P_0)$ and 2-faces in P_t are congruent. Then $P_0 \simeq P_1$.*

Example 1.2 (Bricard's octahedron). Draw four chords on a circle, with 2 intersecting. Now, in the z direction, put a vertex above and below the center of the circle. Now connect the vertices with edges to form 8 faces that intersect each other. If you push the north pole and the south pole towards each other, the polygon is flexible. So this is a non-example to Cauchy's theorem because it is self-intersecting.

Are all non-examples self intersecting?

Theorem 1.3 (Conelly, 1977)**.** *There exists a flexible polyhedral sphere embedded into* $\mathbb{R}^3$*.*

Scientific American used to publish paper cutouts of these kinds of things, where you could build your own flexible polyhedron. Probably dozens of kids made their own flexible polyhedra.[1]

1.3 Spherical Cauchy and high-dimensional Cauchy

Theorem 1.4 (spherical Cauchy's theorem)**.** *For all* $P, P' \subseteq S^3_+$*, the conclusions of Cauchy's theorem hold.*

Proof. The part in our proof where we used a property of Euclidean space was that intersecting a small sphere with a cone gives us a spherical polygon. This is even more clear for spherical polygons. □

Why do we care about spherical polytopes?

Theorem 1.5 (high-dimensional Cauchy)**.** *For all convex polytopes* $P, P' \subseteq \mathbb{R}^d$ *with* $d \geq 3$*,* $\dim(F) = d - 1$*.*

Proof. Prove high-dimensional spherical Cauchy by induction. Then we get this theorem by reduction to the non-spherical case. □

1.4 Rigidity

If you've ever been to a construction site, you know that the rigidity of a building is only dependent on the beams holding up the building.[2] These are the edges. If we have n vertices of a polytope, and we triangulate it, we get $3n - 6$ edges. We want to say that the lengths of these edges should really determine the polytope. Next time, we will prove Dehn's theorem, which talks about this.

[1] According to Professor Pak, you have to be a very special kid to enjoy this sort of thing.
[2] Who knew that discrete geometry would be interesting to engineers?